

Impossibility of the Efimov effect for p -wave interactions

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Whether the Efimov effect is possible, in principle, for p -wave or higher partial-wave interactions is a fundamental question. Recently, there has been a claim that three nonrelativistic particles with resonant p -wave interactions exhibit the Efimov effect. We point out that the assumed p -wave scattering amplitude inevitably causes a negative probability. This indicates that the Efimov states found there cannot be realized in physical situations. We also restate our previous argument that the Efimov effect, defined as an infinite tower of universal bound states characterized by discrete scale invariance, is impossible for p -wave or higher partial-wave interactions.

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When nonrelativistic particles attract with resonant s -wave interactions, three particles always exhibit the Efimov effect, i.e., an infinite tower of universal bound states characterized by discrete scale invariance [1]. Since the Efimov effect is one of the most beautiful phenomena in few-body physics, it is fundamentally important to ask whether the Efimov effect is possible, in principle, for p -wave or higher partial-wave interactions.

Recently, there has been a claim that three nonrelativistic particles with resonant p -wave interactions exhibit the Efimov effect [2]. Their analysis is based on the assumption that the effective range approximation for the p -wave scattering amplitude,

$$f_p(k, \theta) = \frac{k^2 \cos \theta}{-\frac{1}{a_p} + \frac{1}{2}r_p k^2 - ik^3}, \quad (1)$$

is valid up to $k \rightarrow \infty$ with fixed a_p and r_p , or more formally, for $k < \Lambda$ with $\Lambda \gg |a_p|^{-1/3}, |r_p|$. We point out that this assumption inevitably causes a negative probability regardless of underlying interactions. This indicates that their assumption and, thus, conclusions cannot be realized in physical situations. Our points also apply to the earlier work in Ref. [3].

Non-normalizability of the wave function

In terms of the wave function, the use of Eq. (1) up to $k \rightarrow \infty$ is equivalent to imposing the condition that the singular behavior of the wave function for a small separation r between two interacting particles,

$$\Psi(\mathbf{r}) \propto Y_1^m(\hat{\mathbf{r}}) \frac{1}{r^2} + \cdots, \quad (2)$$

extends to $r \rightarrow 0$. Obviously, such a wave function is too singular to be normalized. The same is true for the three-particle wave functions of the Efimov states claimed in Refs. [2, 3].

Negative probability in the potential

To gain further insight into the normalization issue, we regularize the short-distance behavior by a finite-range potential that is nonzero at $r < R$ and otherwise vanishes. When a two-particle bound state with binding energy $E = -\hbar^2 \kappa^2/m$ exists, its normalized wave function outside the potential range is given by [4, 5]

$$\Psi(\mathbf{r})|_{r>R} = \frac{Y_1^m(\hat{\mathbf{r}})}{\sqrt{-\frac{r_p}{2} - \frac{3\kappa}{2}}} \frac{\partial}{\partial r} \left(\frac{e^{-\kappa r}}{r} \right). \quad (3)$$

References [4, 5] obtained the constraint on the effective range parameter near the resonance $\kappa R \simeq 0$ by requiring that the probability outside the potential should not exceed 1; $\int_{r>R} d\mathbf{r} |\Psi(\mathbf{r})|^2 \simeq -2/(r_p R) \leq 1$. The same constraint can be obtained by a different approach [6].

On the other hand, if the short-range potential $R \rightarrow 0$ equivalent to the large cutoff $\Lambda \rightarrow \infty$ is considered with fixed a_p and r_p , the probability outside the potential exceeds 1. As a consequence, the normalization of the wave function is ensured only by a negative probability in the potential; $\int_{r<R} d\mathbf{r} |\Psi(\mathbf{r})|^2 < 0$, which is obviously unphysical. Similarly, in two-channel models of the Feshbach resonance [5], the occupation probability of the closed-channel molecule becomes negative in the same limit. Therefore, for model interactions studied in Refs. [4, 5], the absence of negative probabilities unexceptionally requires the lower bound on $|r_p| \geq O(1)/R$ near the resonance $|a_p|^{1/3} \gg R$.

Violation of the unitarity bound

We now generally argue that the above negative probability is inevitable regardless of underlying interactions as long as Eq. (1) is assumed with $\Lambda \gg |a_p|^{-1/3}, |r_p|$. Reference [7] showed that scaling dimensions of local operators are bounded from below by requiring that norms of states are non-negative. However, for an effective field

theory incorporating Eq. (1) [8],

$$\mathcal{L} = \psi^\dagger \left(i\hbar\partial_t + \frac{\hbar^2\nabla^2}{2m} \right) \psi - \phi^\dagger \left(i\hbar\partial_t + \frac{\hbar^2\nabla^2}{4m} + \nu \right) \phi + g\phi^\dagger \cdot \psi \overleftrightarrow{\nabla} \psi + g\psi^\dagger \overleftrightarrow{\nabla} \psi^\dagger \cdot \phi \quad (4)$$

(note the unconventional sign in front of the second term), the two-particle operator $\phi \sim \psi \overleftrightarrow{\nabla} \psi$ turns out to have the scaling dimension $\Delta = 1$, which violates the unitarity bound $\Delta \geq 3/2$ [7]. Therefore, the negative-norm state must exist in the two-particle sector of the effective field theory (4).

The existence of a negative-norm state or negative probability indicates that no physical interaction can realize Eq. (1) for $k < \Lambda$ with $\Lambda \gg |a_p|^{-1/3}, |r_p|$ [9]. This, in turn, means that the scaling region $|a_p|^{-1/3}, |r_p| \ll k \ll \Lambda$, in which the p -wave scattering amplitude becomes scale invariant and accordingly the Efimov states appear [2], does not exist. Therefore, the Efimov states claimed in Refs. [2, 3] are not physically realizable bound states but mirages seen by artificially assuming Eq. (1) with $\Lambda \gg |a_p|^{-1/3}, |r_p|$ which is never achieved by physical interactions [10]. Since this is the fundamental issue, further discussion is worthwhile.

Finally, we take this opportunity to restate our argument from Ref. [11] that the Efimov effect is impossible for p -wave or higher partial-wave interactions. Recall that the Efimov effect is defined as the formation of an infinite tower of universal bound states characterized by discrete scale invariance. In order for three-particle systems to exhibit the discrete scale invariance, it is neces-

sary that the two-particle interactions are scale invariant because the symmetry can be broken by the quantum anomaly but cannot be enhanced. By setting the scattering length to be infinite and all effective range parameters to be zero, the ℓ th partial-wave scattering amplitude becomes scale invariant. However, as we have pointed out here, such an interaction for any $\ell \geq 1$ causes the non-normalizability of the wave function or a negative probability if the normalization is imposed. Therefore, as long as we respect principles of quantum mechanics, the Efimov effect is impossible for p -wave or higher partial-wave interactions. This observation led the authors of Ref. [11] to the idea of mixed dimensions in their search for new systems exhibiting Efimov physics.

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Note added

Recently, it was explicitly shown that the p -wave scattering amplitude (1), at the resonance $1/a_p = 0$, contains a contribution of the negative-norm state with binding energy $E = -\hbar^2 r_p^2/(4m)$ [12]. This means that the validity of Eq. (1) has to be taken as $k < \Lambda$ with $\Lambda \ll |r_p|$, which is consistent with the conclusion of this article.

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- [1] V. Efimov, “Energy levels arising from resonant two-body forces in a three-body system,” Phys. Lett. B **33**, 563 (1970).
- [2] E. Braaten, P. Hagen, H.-W. Hammer, and L. Platter, “Efimov effect for p -wave interactions,” arXiv:1110.6829v2 [cond-mat.quant-gas].
- [3] J. H. Macek and J. Sternberg, “Properties of pseudopotentials for higher partial waves,” Phys. Rev. Lett. **97**, 023201 (2006).
- [4] L. Pricoupenko, “Modeling interactions for resonant p -wave scattering,” Phys. Rev. Lett. **96**, 050401 (2006) [arXiv:cond-mat/0505448 (cond-mat.other)]; L. Pricoupenko, “Pseudopotential in resonant regimes,” Phys. Rev. A **73**, 012701 (2006) [arXiv:cond-mat/0508120 (cond-mat.other)].
- [5] M. Jona-Lasinio, L. Pricoupenko, and Y. Castin, “Three fully polarized fermions close to a p -wave Feshbach resonance,” Phys. Rev. A **77**, 043611 (2008) [arXiv:0708.0610 (cond-mat.other)].
- [6] H.-W. Hammer and D. Lee, “Causality and universality in low-energy quantum scattering,” Phys. Lett. B **681**, 500 (2009) [arXiv:0907.1763 (nucl-th)]; H.-W. Hammer and D. Lee, “Causality and the effective range expansion,” Ann. Phys. **325**, 2212 (2010) [arXiv:1002.4603 (nucl-th)].
- [7] Y. Nishida and D. T. Son, “Unitary Fermi gas, ϵ expansion, and nonrelativistic conformal field theories,” in *The BCS-BEC Crossover and the Unitary Fermi Gas*, edited by W. Zwerger, Lecture Notes in Physics (Springer, Berlin, 2012) [arXiv:1004.3597 (cond-mat.quant-gas)]. See Section 3.2 in Chapter 7.
- [8] C. A. Bertulani, H.-W. Hammer, and U. van Kolck, “Effective field theory for halo nuclei: shallow p -wave states,” Nucl. Phys. A **712**, 37 (2002) [arXiv:nucl-th/0205063]; P. F. Bedaque, H.-W. Hammer, and U. van Kolck, “Narrow resonances in effective field theory,” Phys. Lett. B **569**, 159 (2003) [arXiv:nucl-th/0304007].
- [9] We recall that negative-norm states also appear in effective field-theory descriptions of s -wave scatterings with large scattering length and positive effective range $r_s > 0$ [13]. In analogy with the p -wave case in the text, this indicates that no physical interaction can realize the effective range approximation to the s -wave scattering amplitude,
- $$f_s(k) = \frac{1}{-\frac{1}{a_s} + \frac{1}{2}r_s k^2 - ik},$$
- that is valid for $k < \Lambda$ with $|a_s|, r_s \gg \Lambda^{-1} \sim R$. Therefore, $r_s \sim R$ or less is required near the resonance

- $|a_s| \gg R$. This general consequence can be shown rigorously for energy-independent finite-range potentials [14].
- [10] Our argument strictly states that, at the resonance $1/a_p = 0$, $|r_p|$ cannot be arbitrarily smaller than Λ , which is sufficient to exclude the infinite tower of bound states claimed in Refs. [2, 3]. On the other hand, it is logically possible that there is a physically sensible model interaction, although not yet known, in which $|r_p|$ becomes smaller than Λ by a model-dependent numerical factor. Since the binding-energy ratio can be made arbitrarily close to 1 by tuning the mass ratio to 0 [2], one may think that a part of “Efimov mirages” seen in Refs. [2, 3] survives as physical bound states. However, in the absence of a physically sensible limit to achieve the discrete scale invariance over a wide range of momenta, it is inappropriate to regard them as Efimov states.
- [11] Y. Nishida and S. Tan, “*Liberating Efimov physics from three dimensions*,” *Few-Body Syst.* **51**, 191 (2011) [arXiv:1104.2387 (cond-mat.quant-gas)].
- [12] E. Braaten, P. Hagen, H.-W. Hammer, and L. Platter, “*Renormalization in the three-body problem with resonant p-wave interactions*,” *Phys. Rev. A* **86**, 012711 (2012) [arXiv:1110.6829 (cond-mat.quant-gas)].
- [13] E. Braaten, M. Kusunoki, and D. Zhang, “*Scattering models for ultracold atoms*,” *Ann. Phys.* **323**, 1770 (2008) [arXiv:0709.0499 (cond-mat.other)].
- [14] D. R. Phillips and T. D. Cohen, “*How short is too short? Constraining zero-range interactions in nucleon-nucleon scattering*,” *Phys. Lett. B* **390**, 7 (1997) [arXiv:nucl-th/9607048].